Solutions of Linear Planar Systems

Consider $\frac{dX}{dt} = A\,X(t)$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ with a, b, c, and d real numbers. Let λ_1 and λ_2 be the eignevalues of A with the corresponding eigenvectors V_1 and V_2, respectively.

1. λ_1 and λ_2 are real-valued and $\lambda_1 \neq \lambda_2$. The general solution is $X(t) = c_1\,e^{\lambda_1 t}\,V_1 + c_2\,e^{\lambda_2 t}\,V_2$, where c_1 and c_2 are constants.

2. $\lambda = \lambda_1 = \lambda_2$.

   (a) V_1 and V_2 are linearly independent. The general solution is $X(t) = e^{\lambda t}\,V$, where V is any vector in $\Re^2$.

   (b) V_1 and V_2 are not linearly independent. Let V be V_1 or V_2 and let U be a solution of the matrix equation $(A - \lambda I)U = V$. The general solution is $X(t) = c_1\,e^{\lambda t}\,V + c_2\,e^{\lambda t}\,(tV + U)$, where c_1 and c_2 are constants.

3. λ_1 and λ_2 are complex-valued: $\lambda_1,\,\lambda_2 = \alpha \pm i\beta$ with $\beta \neq 0$. The general solution is $X(t) = c_1\,\mathrm{Re}(e^{\lambda_1 t}\,V_1) + c_2\,\mathrm{Im}(e^{\lambda_1 t}\,V_1)$, where $\mathrm{Re}(*)$ and $\mathrm{Im}(*)$ are the real and imaginary parts of $*$, respectively, and c_1 and c_2 are constants.